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Linear versus nonlinear time series analysis—smoothed correlation integrals

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Dedicated to Cor M. van den Bleek on the occasion of his retirement

Abstract

The classical analysis of stationary time series is based on the study of autocovariances and spectra. This type of analysis is especially suitable for Gaussian time series. After it became known that also nonlinear deterministic systems can behave in a seemingly random (chaotic) way, methods were developed to detect such nonlinear (and deterministic) sources. These methods are to a large extend based on the use of correlation integrals. Though it is known that these two methods of analysis provide information which is in some sense complementary, not much is known about the possible relations between the information they provide. In this paper we investigate the correlation integrals, and the quantities which can be derived from them, of Gaussian time series in terms of their autocovariances and spectra.

Keywords: Autocovariances; Power spectrum; Correlation integrals; Dimension; Entropy

1. Introduction

During the past years I have been cooperating with van den Bleek and his group in attempts to apply chaos theory to the study of fluidized beds. In particular, the so-called nonlinear time series analysis in terms of correlation integrals turned out to be of importance. Apart from various practical questions, see [1], this work also provoked theoretical questions. The present paper grew out of an attempt to understand the relation between the Shannon–Kolmogorov entropy and the power spectrum as given in [2] and used in the investigation of pressure signals of fluidized beds. We give here, for Gaussian time series, a complete description of the (smoothed) correlation integrals in terms of the power spectrum (or the autocovariances). It turns out that the results imply that the Shannon–Kolmogorov entropy and the correlation entropy are quite different notions.

Time series analysis provides methods to characterize the dynamical regime of a system on the basis of the fluctuations of a relevant quantity as a function of time. From the mathematical point of view the task is to extract from a time series quantities which may provide useful information and to study the relations between such quantities.

A first class of such quantities consists of the autocovariances and the related power spectrum (definitions will be given below). These quantities give a complete characterization of a time series provided that the time series is generated by a linear (stochastic) Gaussian process. The analysis in terms of these quantities is called linear time series analysis, e.g. see [3].

Later it was observed that also time series which are aperiodic and even apparently unpredictable could be generated by completely deterministic processes. The question whether or not there is a deterministic structure behind such a time series cannot be decided on the basis of the autocovariances. For this purpose new quantities where introduced, see [4]. The most important ones are based on correlation integrals. These methods are the content of nonlinear or chaotic time series analysis, e.g. see [5]. For a survey of applications of these nonlinear methods to the analysis of multiphase reactors, see [1].

A refinement of this notion, the smoothed correlation integral, was introduced (without using this term) first in a test for reversibility, see [6], and later in order to improve a general test, originally due to Kantz, for distinguishing different types of time series, see [7]. It turns out that for time series, which are generated by a linear Gaussian process, these smoothed correlation integrals can be related in a rather direct way to autocovariances. These relations are the main subject of this paper.

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Nomenclature

B_k	covariance matrix of the reconstruction measure μ_k
$C^k(\varepsilon)$	k-dimensional correlation integral at distance ε
$d(\varepsilon_0)$	dimension per unit time at length scale ε_0
D	dimension
$D(\varepsilon_0, k)$	dimension at length scale ε_0 and embedding dimension k
f	function, defined by $f(s) = e^{-s^2/2}$
F	read out map
$G_\varepsilon(\lambda)$	function defined by $G_\varepsilon(\lambda) = (1/2)\ln(\varepsilon^2/(2\lambda + \varepsilon^2))$
h	function, defined by $h(s) = 1$ for $s \le 1$ and $h(s) = 0$ for $s > 1$
H	entropy
$H(\varepsilon_0)$	entropy at length scale ε_0
$H(\varepsilon_0, k)$	entropy at length scale ε_0 and embedding dimension k
$\mathrm{sd}(\varepsilon_0)$	smoothed dimension per unit time at length scale ε_0
$\mathrm{SC}^k(\varepsilon)$	smoothed k-dimensional correlation integral at distance ε
$\mathrm{SH}(\varepsilon_0)$	smoothed entropy at length scale ε_0
$X(n)$	nth element of the time series X
$X_n^{(2m+1)}$	nth $(2m+1)$-dimensional reconstruction vector of the time series X

Greek letters

λ_i^k	ith eigenvalue of B_k
μ_k	k-dimensional reconstruction measure
ρ_k	kth autocovariance
Φ	power spectrum
$\varphi : M \to M$	time evolution map of a dynamical system with state space M
ω	(angular) frequency

We now give the definitions of the various notions which occurred above.

Time series and stationarity. From the mathematical point of view we need time series which are defined ‘for all time’. The time itself we assume to be discrete: this is just for simplicity and this simplicity is harmless because when recording a time series one has to use a discretized time anyway. So a time series is a function X which assigns to each time $n \in \mathbf{N}$ a value $X(n)$ which we assume to be a real number ($\mathbf{N}$ denotes the set of natural numbers $0, 1, 2, \ldots$). An important property which a time series may or may not have is *stationarity*. We say that a time series is stationary (in the strict sense) if, for each k, there is a probability measure μ_k on $\mathbf{R}^k$ which describes the density of the k-dimensional *reconstruction vectors* $\{(X(m), \ldots, X(m+k-1))\}_m$, or, more formally, if for each non-negative continuous function $g : \mathbf{R}^k \to \mathbf{R}$ we have

$$\int_{\mathbf{R}^k} g\, d\mu_k = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(X(i), \ldots, X(i+k-1)). \qquad (1)$$

In this case μ_k is called the k-dimensional *reconstruction measure*. Averages, as on the right-hand side of the above equation, will also be denoted by $\mathcal{E}_i(g(X(i), \ldots, X(i+k-1)))$. Since measured time series never have infinite length, we should interpret them as (randomly situated) intervals $\{X(n)\}_{n=n_0}^{n=n_1}$ from such time series. Just as one may ‘conclude’, on a statistical basis, that the members of a population A are in average bigger than members of a population B on the basis of (sufficiently large) samples from these populations, one may also ‘conclude’, on a statistical basis, that two (sufficiently long) segments of time series belong to stationary time series whose reconstruction measures are significantly different.

Unless explicitly stated otherwise we assume that our time series will be stationary and in average zero, i.e. that $\mathcal{E}_i(X(i)) = 0$.

Autocovariances and power spectrum. For a time series $\{X(n)\}$ the kth *autocovariance* ρ_k is $\mathcal{E}_n(X(n)X(n+k))$ (for $k = 0$ this is also called the variance). Note that $\rho_k = \rho_{-k}$. In terms of these autocovariances the *power spectrum* can be given as

$$\Phi(\omega) = \sum_{k=-\infty}^{\infty} \rho_k\, e^{-ik\omega}. \qquad (2)$$

Note that $\Phi(\omega) = \Phi(-\omega)$ and $\Phi(\omega) = \Phi(\omega + 2\pi)$. (We ignore here the problems related with the possibility that the sum may not converge and that therefore Φ must be interpreted as a generalized function.)

Correlation integrals. For a stationary time series with reconstruction measures μ_k the k-dimensional *correlation integral at distance* ε, $C^k(\varepsilon)$, is defined as the $(\mu_k \times \mu_k)$ measure of the set $\{(x, y) \in \mathbf{R}^k \times \mathbf{R}^k | \mathrm{dist}(x, y) \le \varepsilon\}$. Here the distance $\mathrm{dist}(x, y)$ is the maximum distance $\max_{i=1,\ldots,k}|x_i - y_i|$. If we define $h(s)$ as the real function which is 1 for $s \le 1$ and 0 for $s > 1$, then

$$C^k(\varepsilon) = \int_{\mathbf{R}^k\times\mathbf{R}^k} h\left(\frac{\mathrm{dist}(x, y)}{\varepsilon}\right) d\mu_k(x)\, d\mu_k(y). \qquad (3)$$

We obtain the smoothed correlation integrals by replacing the discontinuous function $h(s)$ by the smooth function:

$$f(s) = e^{-s^2/2}. \qquad (4)$$

Note that here also we get numbers between 0 and 1; for smaller values of ε the integral is ‘concentrated’ on a smaller neighborhood of the diagonal in $\mathbf{R}^k \times \mathbf{R}^k$. These smoothed correlation integrals are denoted by $\mathrm{SC}^k(\varepsilon)$.

2. Smoothed correlation integrals of time series generated by deterministic systems

We first give a short summary of the properties of correlation integrals and then relate them to the smoothed correlation integrals.

Unless explicitly stated otherwise, we consider in this section only time series which are generated by smooth deterministic dynamical systems. With this we mean the following:

- there is a finite dimensional state space M, which we may assume to be a vector space—the dimension of M is denoted by m;
- there is an invertible map $\varphi : M \to M$ which determines the time evolution in the system, i.e. if we have the state $x \in M$ at time n, then we have the state $\varphi(x)$ at time $n+1$ (for the case that φ is not invertible, see [8]);
- there is a read out function $F : M \to \mathbf{R}$ which assigns to each state $x \in M$ the value $F(x)$ which is recorded when the system is in the state x.

So with an initial state x_0 there corresponds an orbit $\mathcal{O}(x_0) = (x_0, x_1 = \varphi(x_0), x_2 = \varphi^2(x_0), \dots)$ and to this orbit corresponds a time series $X(i) = F(x_i)$.

For technical reasons we have to assume that that both φ and F are at least once continuously differentiable and that for each initial point x_0, the corresponding orbit $\mathcal{O}(x_0)$ is bounded. According to the reconstruction theorem, see [4], almost all pairs (φ, F) define *observable systems* in the sense that each state $x \in M$ of the dynamical system is uniquely determined by the sequence of $2m + 1$ successive measurements which one obtains if the systems starts in x, i.e. by $(F(x), F(\varphi(x)), \dots, F(\varphi^{2m}(x)))$. Also, if $\tilde{F}$ is another read out function, such that $(\varphi, \tilde{F})$ also defines an observable system and if X and $\tilde{X}$ are the time series corresponding to the same initial state x_0 for the read out functions F and $\tilde{F}$, respectively, then there is a strong relation between recursion in the two time series in the sense that for some constant $K > 1$ and for any $i, j > 0$ one has

$$K^{-1}\|X_i^{(2m+1)} - X_j^{(2m+1)}\| < \|\tilde{X}_i^{(2m+1)} - \tilde{X}_j^{(2m+1)}\| < K\|X_i^{(2m+1)} - X_j^{(2m+1)}\|, \quad (5)$$

where $X_n^{(2m+1)}$ and $\tilde{X}_n^{(2m+1)}$ denote the $(2m+1)$-dimensional reconstruction vector of X, respectively, $\tilde{X}$ starting with its nth element. This gives an indication why one is interested in norms of differences of reconstruction vectors and in the correlation integrals which describe the statistics of these distances. In fact, one can prove that the following quantities

$$D = \lim_{k\to\infty} \lim_{\varepsilon\to 0} \frac{\ln(C^k(\varepsilon))}{\ln(\varepsilon)} \quad (6)$$

and

$$H = \lim_{\varepsilon\to 0} \lim_{k\to\infty} -\frac{\ln(C^k(\varepsilon))}{k} \quad (7)$$

are the same, and finite, for time series X and $\tilde{X}$ as above. For this reason we call these quantities *intrinsic* (independent of the read out function as long as one stays within the class of observable systems). These quantities have a clear geometric and dynamical meaning: D is the dimension of the closure of the orbit $\mathcal{O}(x_0)$ and H is a measure for the sensitive dependence on initial state. They are called the *correlation dimension* and the *correlation entropy*, respectively, e.g. see [5]. These quantities, and the correlation integrals on which they are based, became the main tool of what is now called nonlinear time series analysis. For stochastic time series, e.g. for time series generated by autoregressive models these quantities are infinite. However, even for such time series there are approximate quantities like

$$D(\varepsilon_0, k) = \left.\frac{\mathrm{d}\ln(C^k(\varepsilon))}{\mathrm{d}\ln(\varepsilon)}\right|_{\varepsilon=\varepsilon_0} \quad (8)$$

and

$$H(\varepsilon_0, k) = \ln(C^k(\varepsilon_0)) - \ln(C^{k+1}(\varepsilon_0)) \quad (9)$$

which are finite and which can be interpreted as the dimension, respectively the entropy, at length scale ε_0 and embedding dimension k. One can even make this independent of the embedding dimension k by defining

$$d(\varepsilon_0) = \lim_{k\to\infty} \frac{1}{k} D(\varepsilon_0, k), \quad (10)$$

the dimension per unit time at length scale ε_0, and

$$H(\varepsilon_0) = \lim_{k\to\infty} \frac{\ln(C^k(\varepsilon_0))}{k}, \quad (11)$$

the entropy at length scale ε_0. These quantities are still finite for stochastic time series.

The behavior of these quantities, as function of ε_0 (and k), not only show the difference between time series generated by deterministic and stochastic systems but can be used also to distinguish between different dynamical regimes.

In this context it is important to observe that these quantities give information which cannot be extracted from the autocovariances. For this we discuss two examples:

Example 1 (The logistic system). We consider a dynamical system with state space $[-1, +1]$ and whose evolution is determined by the map $\varphi(x) = 1 - 2x^2$ (as a read out function we just take the identity). There is an extensive theory about this system and related systems, for an introductory treatment, see [9]. For almost any initial state, the resulting orbit (or time series) has variance $\rho_0 = 1/2$ while all the other autocovariances $\rho_1, \rho_2, \dots$ are zero. This means that from the autocovariances one cannot deduce that these time series are generated by a deterministic system: it just as well could have been obtained by choosing each element of the time series randomly and independently from a (Gaussian) distribution with mean zero and variance 1/2.

Example 2 (Autoregressive systems). An autoregressive system generates time series according to the following formula:

$$X(n) = a_1X(n-1) + a_2X(n-2) + \cdots + a_kX(n-k) + \varepsilon_n, \quad (12)$$

where the ε_n are chosen independently and randomly from a distribution with mean 0 and positive variance (if this distribution is Gaussian, the corresponding time series is Gaussian, see the next section). The coefficients a_i have to satisfy a condition so that the generated time series do not diverge to infinity. Time series generated by such systems are the standard examples of stochastic time series. Whenever the autocovariances $\rho_0, \rho_1, \dots, \rho_k$ are given, one can produce the coefficients $a_1, \dots, a_k$ such that the autocovariances of the time series generated by the corresponding autoregressive system, provided the variance of the ε_n is right, are exactly $\rho_0, \dots, \rho_k$.

Now we can change such an autoregressive system into a deterministic system without changing the autocovariances of the time series it produces. For this one replaces the ε_n by the successive values produced by the above logistic system (if necessary multiplied by a constant in order to obtain the right variance).

Replacing correlation integrals by smoothed correlation integrals. We now show that, when replacing the correlation integrals in the definitions of correlation dimension and correlation entropy by smoothed correlation integrals, and thus defining the smoothed correlation dimension and entropy, denoted by SD and SH, respectively, we obtain quantities which are still intrinsic and which are finite for time series generated by deterministic dynamical systems. The fact that these smoothed quantities are still intrinsic follows easily from the definition and the inequalities (5).

We will now show that the smoothed correlation dimension and entropy are in fact majorated by the correlation dimension and entropy.

In the definition of the correlation integral $C^k(\varepsilon)$ we have under the integral an expression which can be written as

$$\prod_{i=1}^{k} h\left(\frac{|x_i - y_i|}{\varepsilon}\right), \quad (13)$$

where $h(s)$ is the function which is 1 for $s \leq 1$ and which is 0 for $s > 1$ and where $x_1, \dots, x_k$ and $y_1, \dots, y_k$ are the components of two elements (reconstruction vectors) in $\mathbf{R}^k$.

In the definition of the smoothed correlation integral the function h is replaced by the function $f(s) = e^{-s^2/2}$. In order to compare the two definitions we minorate this function by functions f^δ:

- $f^\delta(s) = 1 - \delta$ if $s \leq d(\delta)$,
- $f^\delta(s) = 0$ if $s > d(\delta)$.

The value of $d(\delta)$ is defined by the equation:

$$e^{-(d(\delta))^2/2} = 1 - \delta. \quad (14)$$

This implies indeed that for all $\delta > 0$ we have $f^\delta \leq f$. Note that, for small values of δ, we have $d(\delta) \sim \sqrt{2\delta}$.

Using that $f^\delta \leq f$ we find the following inequality between correlation integrals and smoothed correlation integrals:

$$SC^k(\varepsilon) \geq (1-\delta)^k C^k(d(\delta)\varepsilon). \quad (15)$$

From the fact that this inequality holds for δ arbitrarily close to 0, it follows that the limits defining the correlation dimension and entropy cannot increase if we replace the correlation integrals by smoothed correlation integrals.

Remark. It is even possible to show that the smoothed correlation dimension equals the usual correlation dimension; for the smoothed correlation entropy we conjecture the same.

3. Smoothed correlation integrals of Gaussian time series

A probability measure μ on $\mathbf{R}^k$ is Gaussian if it has a density of the form

$$p(x) = |B|^{-1/2}(2\pi)^{-k/2}\, e^{-\langle x, B^{-1}x\rangle/2}, \quad (16)$$

where B, the covariance matrix of the distribution, is a symmetric and strictly positive $k \times k$ matrix with determinant $|B|$. We say that a time series is Gaussian if all its reconstruction measures are Gaussian. If a Gaussian time series has autocovariances ρ_k then the covariance matrix B_k of its k-dimensional reconstruction measure is the Toeplitz matrix:

$$B_k = \begin{pmatrix} \rho_0 & \rho_1 & \rho_2 & \cdot & \rho_{k-1} \\ \rho_1 & \rho_0 & \rho_1 & \cdot & \cdot \\ \rho_2 & \rho_1 & \rho_0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \rho_{k-1} & \cdot & \cdot & \cdot & \rho_0 \end{pmatrix}. \quad (17)$$

These and the following formulae can be easily derived from the following two well-known formulae (for symmetric and strictly positive B):

$$\int_{\mathbf{R}^k} e^{-\langle x, B^{-1}x\rangle/2}\, dx = |B|^{1/2}(2\pi)^{k/2} \quad (18)$$

and

$$\int_{\mathbf{R}^k} x_i x_j |B|^{-1/2}(2\pi)^{-k/2}\, e^{-\langle x, B^{-1}x\rangle/2}\, dx = b_{ij}, \quad (19)$$

where b_{ij} is the (i, j)th element of the matrix B.

In this section we want to find expressions for the smoothed correlation integrals of Gaussian time series in terms of the autocovariances. For this we have to consider the product measure $\mu_k \times \mu_k$ on $\mathbf{R}^k \times \mathbf{R}^k$. Denoting the covariance matrix of μ_k by B_k this product measure has density

$$p(x, y) = |B_k|^{-1}(2\pi)^{-k}\,\mathrm{e}^{-(\langle x, B_k^{-1}x\rangle + \langle y, B_k^{-1}y\rangle)/2}. \tag{20}$$

We transform this density to the u, v coordinates which are defined by $u = (x + y)/2$ and $v = x - y$ (note that the determinant of this transformation is -1). Using

$$\langle x, B_k^{-1}x\rangle + \langle y, B_k^{-1}y\rangle = \langle u, (\tfrac{1}{2}B_k)^{-1}u\rangle + \langle v, (2B_k)^{-1}v\rangle \tag{21}$$

we find for the density of $\mu_k \times \mu_k$, with respect to the u, v coordinates:

$$\begin{aligned} p(u, v) &= (|\tfrac{1}{2}B_k|^{-1/2}(2\pi)^{-k/2}\,\mathrm{e}^{-(\langle u,(B_k/2)^{-1}u\rangle)/2}) \\ &\quad\times (|2B_k|^{-1/2}(2\pi)^{-k/2}\,\mathrm{e}^{-(\langle v,(2B_k)^{-1}v\rangle)/2}). \end{aligned} \tag{22}$$

Since, for the calculation of the smoothed correlation integrals, we are only interested in the distribution of the differences of reconstruction vectors, we can summarize the result of the above calculation by saying:

If the k-dimensional reconstruction vectors have a Gaussian distribution μ_k with covariance matrix B_k, then the differences of these reconstruction vectors also have a Gaussian distribution but with covariance matrix $2B_k$.

The k-dimensional smoothed correlation integral at distance ε of a Gaussian time series whose k-dimensional reconstruction measure μ_k has autocovariance matrix B_k equals the integral:

$$\begin{aligned} \mathrm{SC}^k(\varepsilon) &= \int_{\mathbf{R}^k} |2B_k|^{-1/2}(2\pi)^{-k/2}\,\mathrm{e}^{-(\langle v,(2B_k)^{-1}v\rangle)/2} \\ &\quad\times \mathrm{e}^{-\|v\|^2/2\varepsilon^2}\,\mathrm{d}v. \end{aligned} \tag{23}$$

Introducing the matrix

$$B_{k,\varepsilon} = \varepsilon^2(\varepsilon^2(2B_k)^{-1} + \mathrm{Id})^{-1}, \tag{24}$$

where Id is the $k \times k$ identity matrix, we see that the above expression reduces to

$$\begin{aligned} \mathrm{SC}^k(\varepsilon) &= \int_{\mathbf{R}^k} |2B_k|^{-1/2}(2\pi)^{-k/2}\,\mathrm{e}^{-(\langle v,(B_{k,\varepsilon})^{-1}v\rangle)/2}\,\mathrm{d}v \\ &= |B_{k,\varepsilon}|^{1/2}|2B_k|^{-1/2}. \end{aligned} \tag{25}$$

This last expression can be further evaluated in terms of the (positive) eigenvalues of B_k, which we denote by $\lambda_1^k, \ldots, \lambda_k^k$. From the definition of $B_{k,\varepsilon}$ and the fact that all the matrices B_k, Id, and hence $B_{k,\varepsilon}$ have a common orthogonal basis with respect to which they are all on diagonal form, it follows that the ith eigenvalue if $B_{k,\varepsilon}$ equals

$$\frac{\varepsilon^2\lambda_i^k}{\lambda_i^k + (1/2)\varepsilon^2}. \tag{26}$$

This implies that

$$\mathrm{SC}^k(\varepsilon) = |B_{k,\varepsilon}|^{1/2}|2B_k|^{-1/2} = \left(\prod_{i=1}^{k}\frac{\varepsilon^2}{2\lambda_i^k + \varepsilon^2}\right)^{1/2}. \tag{27}$$

For later reference we put this result in a slightly different form using the functions

$$G_\varepsilon(\lambda) = \frac{1}{2}\,\ln\left(\frac{\varepsilon^2}{2\lambda + \varepsilon^2}\right) \tag{28}$$

we find

$$\ln(\mathrm{SC}^k(\varepsilon)) = \sum_i G_\varepsilon(\lambda_i^k). \tag{29}$$

Finally, we observe that though the eigenvalues $\lambda_1^k, \ldots, \lambda_k^k$ are determined by the autocovariances $\rho_0, \ldots, \rho_{k-1}$, there is no simple expression for these eigenvalues in terms of the autocovariances. Still, for $k \to \infty$, there are asymptotic results. These will be used in the next section.

4. Dimensions, entropies, and power spectrum

As in the previous section we assume that we have a Gaussian time series with autocovariances $\rho_0, \rho_1, \ldots$. As we mentioned in Section 1, the power spectrum is then given by

$$\Phi(\omega) = \sum_{k=-\infty}^{\infty} \rho_k\,\mathrm{e}^{-\mathrm{i}k\omega}. \tag{30}$$

There are (asymptotic) relations between the eigenvalues of the (Toeplitz) covariance matrices B_k of the reconstruction measures of the time series, denoted by $\lambda_i^k, \ldots, \lambda_k^k$, and the power spectrum, see [10, Chapter 4]. For our purpose these relations can be stated as:

For any continuos function $G : [0, \infty) \to \mathbf{R}$, we have

$$\lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} G(\lambda_i^k) = \int_{-\pi}^{\pi} G(\Phi(\omega))\,\mathrm{d}\omega. \tag{31}$$

For this result to be valid one has to assume that the power spectrum is a bounded Lebesgues integrable function. This is certainly the case if the time series is generated by a stochastic autoregressive model, but not if the time series is (quasi) periodic.

On the basis of this result we have, for $k \to \infty$,

$$\frac{1}{k}\,\ln(\mathrm{SC}^k(\varepsilon)) \sim \int_{-\pi}^{\pi} G_\varepsilon(\Phi(\omega))\,\mathrm{d}\omega, \tag{32}$$

where G_ε is as defined in the previous section. This means that we find for the smoothed entropy ‘at length scale ε_0’

$$\begin{aligned} \mathrm{SH}(\varepsilon_0) &= -\lim_{k\to\infty} \frac{1}{k}\,\ln(\mathrm{SC}^k(\varepsilon_0)) \\ &= -\int_{-\pi}^{\pi} \frac{1}{2}\,\ln\left(\frac{\varepsilon_0^2}{2\Phi(\omega) + \varepsilon_0^2}\right)\mathrm{d}\omega. \end{aligned} \tag{33}$$

There is also a corresponding result for the dimension per unit time at length scale ε_0. For calculating the smoothed version of $D(\varepsilon_0, k)$ we need the function

$$\frac{\mathrm{d}G_\varepsilon}{\mathrm{d}\,\ln(\varepsilon)} = \frac{2\lambda}{2\lambda + \varepsilon^2}. \tag{34}$$

This means that the smoothed dimension per unit time, at distance ε_0, equals

$$\mathrm{sd}(\varepsilon_0) = \int_{-\pi}^{\pi} \frac{2\Phi(\omega)}{2\Phi(\omega) + \varepsilon_0^2}\,\mathrm{d}\omega. \tag{35}$$

A final question is the following. For Gaussian time series, the autocovariances (or the power spectrum) contain all the relevant information. What about the (smoothed) correlation integrals. It turns out that the (smoothed) correlation integrals are missing important information. A simple example to see this is the following. Take any stationary Gaussian time series $X = X(0), X(1), \dots$. Now we obtain a second time series $\tilde{X} = \tilde{X}(0), \tilde{X}(1), \dots$ by changing the sign of all the values $X(i)$ with i even, i.e. $X(i) = \tilde{X}(i)$ for i odd and $X(i) = -\tilde{X}(i)$ for i even. Then also $\tilde{X}$ is stationary. And, as one can see easily, the (smoothed) correlation integrals for X and $\tilde{X}$ are the same. Still for the autocovariances ρ_i and $\tilde{\rho}_i$ of these time series we have $\rho_i = \tilde{\rho}_i$ for i even and $\rho_i = -\tilde{\rho}_i$ for i odd. This implies that we have for the power spectra Φ and $\tilde{\Phi}$:

$$\Phi(\omega) = \tilde{\Phi}(\omega \pm \pi). \tag{36}$$

This relation, and our formulae for SH and sd, also imply that the values of $\mathrm{SH}(\varepsilon)$ and $\mathrm{sd}(\varepsilon)$ are the same for both time series. This same observation is also valid when using the usual (non-smoothed) correlation integrals, even if we use instead of the 'maximal distance' the Euclidean or the l_p distance.

Remark. The smoothed entropy $\mathrm{SH}(\varepsilon)$ should not be confused with another notion of ε-entropy, also called rate distortion function, introduced by Shannon and Kolmogorov. The way in which this rate distortion function is related to the power spectrum is completely different, see [2].

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